WNE Linear Algebra
Resit Exam

28 February 2020

Questions A

Question 1.
Is matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ positive definite for all $a, b, c, d > 0$?

Answer 1.
No, for example

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

is not positive definite (it is indefinite). Let

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

then

$$v^\mathsf{T} A v = 1 > 0, \quad w^\mathsf{T} A w = -3 < 0.$$

Question 2.
If $v = (1, 1, 0), w = (-1, 1, 2) \in, \mathbb{R}^3,\ V = \text{lin}(v)$, is the image of vector $w \in \mathbb{R}^3$ under the (linear) orthogonal symmetry about the subspace $V \subset \mathbb{R}^3$ equal to

$$S_V(w) = (1, -1, -2)?$$

Answer 2.
Yes, since vector $v \in V$ is an orthogonal basis of V

$$P_V(w) = \frac{w \cdot v}{v \cdot v} v = \frac{0}{2}(1, 1, 0) = \mathbf{0}.$$

Moreover

$$S_V(w) = 2P_V(w) - w,$$

therefore

$$S_V(w) = -w = (1, -1, -2).$$

Question 3.
If $A \in M(2 \times 2; \mathbb{R})$ and $A + A^\mathsf{T} = \mathbf{0}$, does it follow that $A^\mathsf{T} A = A A^\mathsf{T}$?

Answer 3.
Yes, if $A^\mathsf{T} = -A$ then $A^\mathsf{T} A = (-A)A = -A^2$ and $AA^\mathsf{T} = A(-A) = -A^2$.
Alternatively, if $A^\mathsf{T} = -A$ then

$$A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix},$$

for some $a \in \mathbb{R}$ and

$$AA^\mathsf{T} = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix},$$

$$A^\mathsf{T} A = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix}.$$

Question 4.
Is it possible that $A, B \in M(2 \times 2; \mathbb{R})$, $\det\big(A^2 + 2AB\big) \neq 0$ and $\det A = 0$?

Answer 4.
No, because

$$\det\big(A^2 + 2AB\big) = \det(A(A + 2B)) = \det A \det(A + 2B).$$

Question 5.
Is it possible that $\mathcal{A}, \mathcal{B}$ are two bases of $\mathbb{R}^2$ and

$$M(\mathrm{id})_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}?$$

Answer 5.
No, the matrix $M(\mathrm{id})_{\mathcal{A}}^{\mathcal{B}}$ is invertible for any two bases $\mathcal{A}, \mathcal{B}$ (since $M(\mathrm{id})_{\mathcal{B}}^{\mathcal{A}} M(\mathrm{id})_{\mathcal{A}}^{\mathcal{B}} = M(\mathrm{id})_{\mathcal{A}}^{\mathcal{A}} = I$). Matrix $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ is not invertible since its determinant is equal to 0.
Alternatively, if $\mathcal{A} = (v_1, v_2)$ and $\mathcal{B} = (w_1, w_2)$ then the condition

$$M(\mathrm{id})_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

implies $v_1 = w_1 + w_2,\ v_2 = \mathbf{0}$ which is not possible (vectors of a basis are linearly independent).

Question 6.
Are the affine subspaces $E,\ H \subset \mathbb{R}^3$ given by

$$E\colon \begin{cases} x_1 - x_2 = 5 \\ 2x_2 - x_3 = 6 \end{cases},$$

$$H = (-1, 0, 2) + \mathrm{lin}((1, 1, 2)),$$

parallel?

Answer 6.
Yes, they are.

$$\vec{E}\colon \begin{cases} x_1 - x_2 = 0 \\ 2x_2 - x_3 = 0 \end{cases},$$

$$\vec{E}\colon \begin{cases} x_1 = x_2 \\ x_3 = 2x_2 \end{cases}, \quad x_2 \in \mathbb{R}$$

therefore

$$\vec{E} = \{(x_2, x_2, 2x_2) \in \mathbb{R}^3 \mid x_2 \in \mathbb{R}\} =$$

$$= \{x_2(1, 1, 2) \in \mathbb{R}^3 \mid x_2 \in \mathbb{R}\} = \mathrm{lin}((1, 1, 2)) = \vec{H},$$

which, by definition (see Lecture 11) means that E and H are parallel.

Questions B

Question 1.
Is matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ negative definite for all $a, b, c, d < 0$?

Answer 1.
No, for example

$$A = \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}$$

is not negative definite (it is indefinite). Let

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

then

$$v^\intercal A v = -1 < 0, \quad w^\intercal A w = 3 > 0.$$

Question 2.
If $v = (1, 0, 1), w = (1, 2, 3) \in \mathbb{R}^3,\ V = \mathrm{lin}(v)$, is the image of vector $w \in \mathbb{R}^3$ under the (linear) orthogonal projection on the subspace $V^\perp \subset \mathbb{R}^3$ equal to

$$P_{V^\perp}(w) = (-1, 2, 1)?$$

Answer 2.
Yes, since vector $v \in V$ is an orthogonal basis of V

$$P_V(w) = \frac{w \cdot v}{v \cdot v} v = \frac{4}{2}(1, 0, 1) = (2, 0, 2).$$

Moreover

$$w = P_V(w) + P_{V^\perp}(w),$$

therefore

$$P_{V^\perp}(w) = w - P_V(w) = (1, 2, 3) - (2, 0, 2) = (-1, 2, 1).$$

Question 3.
If $A \in M(2 \times 2; \mathbb{R})$ and $A - A^\intercal = \mathbf{0}$, does it follow that $A^\intercal A = AA^\intercal$?

Answer 3.
Yes, if $A = A^\intercal$ then $A^\intercal A = AA^\intercal = A^2$.

Question 4.
Is it possible that $A, B \in M(2 \times 2; \mathbb{R})$, $\det B = 0$ and $\det\big(2AB + B^2\big) \neq 0$?

Answer 4.
No, because

$$\det\big(2AB + B^2\big) = \det((2A + B)B) = \det(2A + B) \det B.$$

Question 5.
Is it possible that $\mathcal{A}, \mathcal{B}$ are two different bases of $\mathbb{R}^2$ and

$$M(\mathrm{id})_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}?$$

Answer 5.
No, if $\mathcal{A} = (v_1, v_2)$ and $\mathcal{B} = (w_1, w_2)$ then the condition

$$M(\mathrm{id})_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

implies $v_1 = w_1,\ v_2 = w_2$ therefore $\mathcal{A} = \mathcal{B}$.

Question 6.
Are the affine subspaces $E,\ H \subset \mathbb{R}^3$ given by

$$E = (1, 1, 2) + \mathrm{aff}((-1, 0, 1), (1, 1, 2), (3, 2, 3)),$$
$$H = (1, -1, 0) + \mathrm{lin}((2, 1, 1)),$$

parallel?

Answer 6.
Yes, they are.

$$\overrightarrow{E} = \mathrm{lin}((1,1,2)-(-1,0,1),(3,2,3)-(-1,0,1)) = \\ = \mathrm{lin}((2,1,1),(4,2,2)) = \mathrm{lin}((2,1,1)) = \overrightarrow{H},$$

which, by definition (see Lecture 11) means that E and H are parallel.